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Wronskian and Casorati determinant representations for Darboux–Pöschl–Teller potentials and their difference extensions

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Abstract
We consider some special reductions of generic Darboux–Crum dressing formulae and of their difference versions. As a matter of fact, we obtain some new formulae for Darboux–Pöschl–Teller (DPT) potentials by means of Wronskian determinants. For their difference deformations (called DDPT-I and DDPT-II potentials) and the related eigenfunctions, we obtain new formulae described by the ratios of Casorati determinants given by the functional difference generalization of the Darboux–Crum dressing formula.

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1. Introduction

The Darboux–Pöschl–Teller equation (we often abbreviate it as the DPT equation) reads as

$$-y'' + \left(\frac{n(n+1)\alpha^2}{\cos^2 \alpha x} + \frac{m(m+1)\alpha^2}{\sin^2 \alpha x}\right) y = \lambda y. \tag{1}$$

Here $\alpha > 0$ and m, n are some positive integers. The DPT equation is obviously invariant with respect to the action of each of the maps: $m \to -m-1$, $n \to -n-1$. The integrability of (1) was proved in 1882 by Darboux [7].

In [7], he proposed two independent methods for solving (1). First, taking $\sin^2(\alpha x)$ as a new independent variable, he reduced (1) to the Gauss hypergeometric equation and expressed the solutions in terms of the hypergeometric functions. Darboux also remarked

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that the functions $y(m,n) := \cos^{n+1}(\alpha x)\sin^{m+1}(\alpha x)$ represent a particular solution of (1) with $\lambda = \alpha^2(m+n+2)^2$. Taking these solutions as generating functions of some sequence of Darboux transformations connecting the potentials with different values of m and n, he obtained a general solution of the same equation in terms of elementary functions. Comparison of these two forms of solution provides a nontrivial case of the reduction of the Gauss hypergeometric function to elementary functions. Almost 50 years later, both the trigonometric DPT equation, considered here, and its hyperbolic version (obtained by replacing α by $\mathrm{i}\alpha$) were independently solved by Pöschl and Teller [25] (in the context of their studies of the quantum theory of two atomic molecules), using the Frobenius series approach. They also arrived at a hypergeometric representation for the solutions to the DPT equation ignoring somehow the elementary function solutions trivially following from the results of Darboux. In the later literature (see [8, 9, 12] for further references and results), the DPT equation was mainly called the Pöschl–Teller equation[4].

We are somehow convinced that the name of Darboux should also be attributed to an equation which was first solved by him[5].

We present here some new representation for the solution to the DPT equation by means of Wronskian determinants. One of the advantages of our formulae is that they provide a one line derivation of the action of the KdV hierarchy to the DPT potentials taken as initial data (see the precise explanations below). The same kind of results is valid for the action of the difference KdV hierarchy (functional difference Toda hierarchy) on the DDPT-I potentials (or respectively on the DDPT-II potentials) although here we skip the details. The connection between the DPT potentials and the 2D Huygens potentials also follows immediately from our results combined with the Berest–Loutzenko formula for generic 2D Huygens potentials. We also explain that DDPT-II potentials represent the special case of Askey–Wilson potentials distinguished by the fact that the related Askey–Wilson functions can be evaluated by means of Casorati determinants composed from the elementary functions.

2. Darboux–Pöschl–Teller equation and Darboux–Crum dressing formula

2.1. How to represent the DPT equation and its solutions by means of Wronskian determinants

Let us recall the following statement known as the Darboux–Crum dressing formula[6].

In the following, $W(f_1, f_2, \dots, f_m)$ denotes a Wronskian determinant of m functions $f_j(x)$, i.e. $W(f_1, f_2, \dots, f_m) = \det \left\| \partial_x^{j-1} f_k(x) \right\|$, $j, k = 1, \dots, m$.

Theorem 2.1. *Suppose that f_j and f, $j = 1, \dots, m$, satisfy the Schrödinger equations*

$$-f_j'' + v(x) f_j = E_j f_j, \qquad -f'' + v(x) f = Ef.$$

[4] In [8, 9], the links between the solutions of the (hyperbolic) DPT equation with zonal spherical functions on some hyperboloids fixed by the choice of m and n were explained using the change of variables close to that used by Darboux [7] for reducing the (trigonometric) DPT equation to the Gauss hypergeometric equation. The DPT equation can also be identified with the BC_1 quantum Calogero–Moser system. Its difference deformations discussed below correspond to the simplest 'relativistic' two-particle quantum Moser–Calogero system. The general N-particle systems of this kind were introduced in [11] by van Diejen.

[5] In fact, Darboux not only solved the DPT equation but also introduced and solved in a very elegant way its 4-integer elliptic extension [5], rediscovered 100 years later by Verdier and Treibich in the context of considering special reductions of the generic finite gap potentials described by the Its–Matveev formula.

[6] The Fredholm determinant representations for the solutions to the DPT equation and various results concerning their links with combinatorics of Young diagrams and representation theory were obtained by van Diejen and Kirillov [8, 9].

Then the function ψ *defined by the formula*

$$\psi(x, E) := \frac{W(f_1, f_2, \dots, f_m, f)}{W(f_1, f_2, \dots, f_m)} \tag{2}$$

satisfies the Schrödinger equation

$$-\psi'' + u\psi = E\psi, \qquad u := v - 2\partial_x^2 \log W(f_1, f_2, \dots, f_m). \tag{3}$$

In the case $m = 1$, this proposition was found and proved by Darboux [7], and in the general case by Crum [4].

Surprisingly, the link between the latter formula and DPT potentials was explained only in 2002 [12].

In order to explain the connection between the Darboux–Crum dressing theorem and DPT equation, we have to answer the following question: how to choose the numbers E_j, $j = 1, \dots, m$, and the functions $f_1, \dots, f_m$, satisfying $f_j'' = E_j f_j$ in order to get (1) from the Schrödinger equation (3) with $v = 0$? After finding the reply to this question, the general solution to (1) will be given by (2). A precise description of the corresponding choice of E_j, $f_j(x)$ is given by the following statement first proved in [12]. In this section, we give a partially different and simpler proof of the same result.

Theorem 2.2. *Let* $v(x) = 0$; m, n $(m \geqslant n)$ *are the same non-negative integers as in (1),* $d := m - n$, $\alpha > 0$ *and* E_p, $f_p(x)$, $p = 1, \dots, m$, *are defined as follows:*

$$\begin{aligned} &E_p := (\alpha a_p)^2, \qquad f_p(x) = \sin \alpha a_p x, \qquad a_p = p, \qquad d \neq 0, \qquad 1 \leqslant p \leqslant m - n, \\ &a_{m-n+j} = m - n + 2j, \qquad n \neq 0, \quad 1 \leqslant j \leqslant n. \end{aligned} \tag{4}$$

Then the following Wronskian representation for the DPT potential $u_{mn}(x)$ *holds*[7]*:*

$$u_{mn}(x) := \frac{n(n+1)\alpha^2}{\cos^2 \alpha x} + \frac{m(m+1)\alpha^2}{\sin^2 \alpha x} = -2\partial_x^2 \log W(f_1, f_2, \dots, f_m). \tag{5}$$

The general solution to (1) is given by (2). Moreover, in this case the Wronskian $W(f_1, \dots, f_m)(m, n, \alpha, x) \equiv W_{mn}$ *can be explicitly computed:*

$$W_{m0}(\alpha) = (-2\alpha)^{\frac{m(m-1)}{2}} \prod_{k=0}^{m-1} k! \cdot \sin^{\frac{m(m+1)}{2}}(\alpha x), \qquad W_{nn}(\alpha) = W_{n0}(2\alpha). \tag{6}$$

$$\begin{aligned} &W_{mn} = c_{mn} \cdot \sin^{\frac{m(m+1)}{2}}(\alpha x) \cos^{\frac{n(n+1)}{2}}(\alpha x), \\ &c_{mn} = (-2\alpha)^{\frac{m(m-1)}{2}} \cdot 2^{n^2} \prod_{p=0}^{d-1} p! \prod_{k=1}^{n} \frac{(d+2k-1)!}{(2k-1)!}(k-1)!, \qquad m > n, \\ &c_{mm} = (-2\alpha)^{\frac{m(m-1)}{2}} \cdot 2^{m^2} \prod_{k=1}^{m-1} k!, \qquad m \geqslant 2. \end{aligned} \tag{7}$$

Example

$$\begin{aligned} W(\sin x, \sin 2x, \sin 4x, \sin 6x, \sin 8x) &= 2^{24} \times 5 \times 7 \times 9 \cos^6(x) \sin^{15}(x) \\ &= 5284\,823\,040 \cos^6(x) \sin^{15}(x). \end{aligned}$$

Hint of the proof. The crucial part of the proof is to demonstrate formulae (6) and (7) which can be done in two steps. First, one has to establish formula (6) (see [12] for details), which

[7] In particular, for the case $m = n$ we have $a_j = 2j$, $j = 1, \dots, n$.

is also enough for calculating W_{nn} because of the relation $W_{n0}(2\alpha) = W_{nn}(\alpha)$. The next part of the proof sketched below is different from that of [12].

Denote by g_j the functions

$$g_j := \sin(\alpha j x), \qquad j \in \mathbb{Z}.$$

Then it is easy to see that

$$W_{mn} = W(g_1, g_2, \ldots, g_{m-n}, g_{m-n+2}, g_{m-n+4}, \ldots, g_{m+n}).$$

We define the sequence of Wronskians $W_{m-k,n}$ of order $m-k$ by the formulae

$$\begin{aligned} W_{m-1,n} &= W(g_1, g_2, \ldots, g_{m-n-1}, g_{m-n+1}, g_{m-n+3}, \ldots, g_{m+n-1}),\\ W_{m-k,n} &= W(g_1, g_2, \ldots, g_{m-n-k}, g_{m-n-k+2}, g_{m-n-k+4}, \ldots, g_{m-k+n}),\\ W_{n,n} &= W(g_2, g_4, \ldots, g_{2n}). \end{aligned}$$

With these notations, it is easy to prove the recursion relation

$$\begin{aligned} W_{m-k,n} = {}& g_1^{m-k}(-2\alpha)^{m-k-1}(m-k-1-n)!\\ &\cdot \prod_{j=1}^{n}(m-k-1-n+2j)\cdot W_{m-k-1,n}, \qquad 0 \leqslant k \leqslant m-n-1. \end{aligned} \tag{8}$$

Applying the recursion relation (8) $(m-n)$ times (for $0 \leqslant k \leqslant m-n-1$), we obtain

$$\begin{aligned} W_{mn} = {}& [\sin(\alpha x)]^{(m-n)(m+n+1)/2}(-2\alpha)^{(m-n)(m+n-1)/2}\\ &\cdot \prod_{j=0}^{m-n-1} j! \prod_{k=1}^{m-n}\prod_{j=1}^{n}(k+2j-1)\cdot W_{nn}. \end{aligned} \tag{9}$$

Substituting the rhs of (6) on the rhs of (9), we get (7) which completes the proof. □

2.2. *The action of the KdV hierarchy on the DPT potentials*

The advantage of the Wronskian representation for the DPT potentials is that it allows us to describe immediately the action of KdV and higher flows of the KdV hierarchy on these potentials taken as initial data. The jth equation of the KdV hierarchy can be obtained as a compatibility condition of (3) and the following evolution equation:

$$\partial_{t_j} f = c_j \partial_x^{2j+1} f - \frac{(2j+1)c_j}{2} u(x,t_j)\partial_x^{2j-1} f + \sum_{p=0}^{2j-2} u_p(x,t_j)\partial_x^p f, \qquad c_j \in \mathbb{R}. \tag{10}$$

In particular for $j = 1$, $c_j = -4$ and $u_0 = 3u_x$, compatibility of (1) and (10) [8] implies that v satisfies the KdV equation:

$$u_t = 6uu_x - u_{xxx}. \tag{11}$$

Then the following formula describes the solution to the jth KdV equation satisfying the initial condition $u(x,0) = u_{mn}(x)$:

$$u(x,t_j) = -2\partial_x^2 \log W(f_1, f_2, \ldots, f_m). \tag{12}$$

$$\begin{aligned} f_p(x,t) &= \sin[\alpha p x + (-1)^j c_j \alpha^{2j+1} p^{2j+1} t_j], && p = 1, \ldots, d, \quad d = m-n,\\ f_{d+l}(x,t) &= \sin[\alpha(d+2l)x + (-1)^j c_j \alpha^{2j+1}(d+2l)^{2j+1} t_j], && l = 1, \ldots, n, \quad m \geqslant n, \end{aligned}$$

[8] In general the choice of the real constants c_j is arbitrary: it just fixes normalization of the KdV hierarchy.

In particular, the solution of (11) with the initial condition (5) is given by formula (12) with

$$f_p(x,t) := \sin(\alpha p x + 4\alpha^3 p^3 t), \qquad p = 1,\ldots,d, \quad d = m-n,$$
$$f_{d+l}(x,t) := \sin[\alpha(d+2l)(x + 4\alpha^2(d+2l)^2 t)], \qquad l = 1,\ldots,n, \quad m \geqslant n.$$

The proof follows immediately from theorem 2.2 and the general theorems concerning the Darboux covariance property of the linear evolution PDE of the form

$$f_t = \sum_{j=0}^{N} u_j(x,t) f^{(j)}(x,t), \qquad f^{(j)} := \partial_x^j f,$$

first formulated in [17]. For the detailed proof and various extensions, see [19].

2.3. *Differential identities*

Formulae (6) and (7) lead to some nontrivial identities. Let $\widehat{D}$ be a differential operator defined as follows:

$$(\widehat{D} f)(x) = \left(\prod_{l=2}^{m} \partial_{x_l}^{l-1}\right) f(x_1, x_2, \ldots, x_m)\Big|_{x_1=x_2=\cdots=x_m=x}. \tag{13}$$

Let D_m be the following determinant: $D_m = \det a_{j,k}$, $a_{j,k} = \sin j x_k$, with $1 \leqslant j,k \leqslant m$.

Clearly, we have

$$\widehat{D} D_n = \det \partial_x^{j-1} \sin kx = W(\sin x, \sin 2x, \ldots, \sin nx) = W_{n0}. \tag{14}$$

The determinant D_n can be easily computed [26]:

$$\det\|\sin(jx_k)\| = 2^{n(n-1)} \prod_{j=1}^{n} \sin x_k \prod_{1\leqslant l<k\leqslant n} \sin\frac{x_l + x_k}{2} \sin\frac{x_l - x_k}{2}$$
$$= 2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} \sin x_k \prod_{n\geqslant k>j\geqslant 1} (\cos x_k - \cos x_j), \qquad j,k = 1,\ldots,n. \tag{15}$$

Now, substituting this expression for D_n in (14) and taking into account formula (7) for W_{n0}, we get the following identity:

$$\left(\prod_{l=2}^{n} \partial_{x_l}^{l-1}\right)\left[\left(\prod_{k=1}^{n} \sin x_k\right)\cdot\left(\prod_{m\geqslant k>j\geqslant 1} (\cos x_j - \cos x_k)\right)\right]\Bigg|_{x_1=x_2=\cdots=x_n=x}$$
$$= \prod_{k=1}^{n-1} k! \cdot \sin^{\frac{n(n+1)}{2}} x, \qquad n \geqslant 2. \tag{16}$$

Further examples of the related differential identities can be found in [12].

3. Casorati determinant approach to the DDPT-I model

Here we discuss a family of integrable deformations of the DPT model, representing the special case of the difference equation[9]

$$v(x) f(x+h) + f(x-h) = \lambda f(x), \qquad h > 0. \tag{17}$$

[9] It is important to mention that multiplying any solution of (17) by any h-periodic function of x, we again obtain the solution of (17). Therefore, in contrast to the lattice case, the space of solutions of (17) is infinite dimensional.

In the following we use the abbreviations:

$c(x) := \cos \alpha x, \qquad s(x) := \sin \alpha x, \qquad \alpha > 0.$

Below we consider the special potentials $v = v_{mn}(x, h, \alpha)$, defined as follows:

$$v_{mn}(x) := \frac{c(x + (n+1)h)c(x - nh)s(x + (m+1)h)s(x - mh)}{c(x)c(x+h)s(x)s(x+h)} \tag{18}$$

$$= v_{n0}(x)v_{0m}(x). \tag{19}$$

Equation (17) with $v(x) = v_{mn}(x)$ is called below the DDPT-I equation.

It has been shown in [22, 23] that the DDPT-I model is integrable and its global solution can be expressed by means of the elementary functions.

In the limit $h \to 0$, the DDPT-1 equation reduces to the differential DPT equation [22, 23]. The case $n = 0$ or $m = 0$ was studied in recent works by Ruisjenaars [10] and van Diejen and Kirillov [27], using different tools and leading to formulae different from ours and having a more complicated combinatorial structure. The lattice specialization of the case $m = 0$, corresponding to the choice $h = 1, x = j, j \in \mathbb{Z}$, was studied by Spiridonov and Zhedanov [28, 29] using the recursive approach 'à la Darboux'.

Remark 3.1. Similar to the DPT equation, the potential $v_{nm}(x)$ is invariant under the action of each of the mappings $m \to -m - 1$ and $n \to -n - 1$.

3.1. Multiple difference Darboux transform and DDPT-I model

First, we recall a simplest case of Matveev's dressing theorem [18–21] for the case of a second-order functional difference equation. For further generalizations and non-Abelian and non-stationary extensions, see [19].

In the following, the notation $\delta_n(x) = \delta_n[f_1, f_2, \ldots, f_n](x)$ will be used for the Casorati determinant:

$$\delta_n(x) = \det A, \qquad A_{ij} := f_j(x - (n + 1 - 2\mathrm{i})h), \qquad i, j = 1, \ldots, n. \tag{20}$$

We also use below the following notations:

$$\phi_n(x) := \frac{\delta_{n+1}[f, f_1, f_2, \ldots, f_n](x)}{\delta_n[f_1, f_2, \ldots, f_n](x+h)}, \qquad n \geqslant 1, \quad \phi_0(x) := f(x), \tag{21}$$

$$\begin{aligned} \Phi_j(x) &= \phi_{j-1}(x)|_{f(x)=f_j(x)}, \qquad j = 1, 2, \ldots, n. \\ \kappa_j(x) &= \frac{\Phi_j(x-h)}{\Phi_j(x+h)}, \end{aligned} \tag{22}$$

where f_j are some fixed solutions to

$$v(x)f_j(x+h) + f_j(x-h) = \lambda_j f_j(x) \tag{23}$$

and f denotes any solution of (17). The function $\phi_n(x)$ defined above is called the *N*-fold (difference), Darboux transform of $f(x)$, generated by $f_1, f_2, \ldots, f_n$.

Theorem 3.1. *The function ϕ_n represents a general solution to the functional difference equation*

$$v_n(x)\phi_n(x+h) + \phi_n(x-h) = \lambda\phi_n(x), \tag{24}$$

$$v_n(x) = v(x+nh)\frac{\delta_n(x-h)\delta_n(x+2h)}{\delta_n(x)\delta_n(x+h)}. \tag{25}$$

In other words, Darboux transformation maps (17) into an equation of the same form, with the same value of spectral parameter λ, but with a new potential constructed in terms of the initial potential $v(x)$ and a fixed solution $f_j(x)$ of (23)[10]. The function $\phi_n(x)$ can also be represented in the following factorized form:

$$\phi_n(x) = (T^{-1} - \kappa_n(x)T)\dots(T^{-1} - \kappa_1(x)T)f(x,\lambda),$$

where T is the shift operator: $T^{\pm k}f(x) = f(x \pm kh)$. For this reason, it is often called the N-fold Darboux transformation of $f(x)$. Below, theorem 3.1 will be applied to the case $v(x) = 1$ in order to show that the potentials $v_{mn}(x)$ allow for a natural representation in a form of the ratio of four Casorati determinants. To prove this, we need to calculate some special Casorati determinants.

3.2. *Casorati addition formula for sine functions*

Let $m = n + d$ where n and d are some non-negative integers, α being a real parameter. Suppose that the functions $f_j(x)$ are defined as in (4). Then they satisfy (23), with $v = 1$ and $\lambda_j = 2\cos(\alpha a_j h)$.

We define the determinant $\delta_m(x,n)$ by the formula

$$\delta_m(x,n) = \det A, \quad A_{ij} := f_j((x - (m+1-2i)h)), \qquad i,j = 1,\dots,m. \tag{26}$$

Theorem 3.2. *The determinant $\delta_m(x,n)$ is equal to the following product*[11]*:*

(a) $m \geqslant 2, 1 \leqslant n \leqslant m-1$

$$\delta_m(x,n) = c_{mn}\cdot\prod_{k=-m+1}^{m-1}\sin^{[\frac{m+1-|k|}{2}]}\alpha(x+kh)\prod_{k=-n+1}^{n-1}\cos^{[\frac{n+1-|k|}{2}]}\alpha(x+kh),$$
$$c_{mn} = 2^{\frac{n(n+1)}{2}}(-4)^{\frac{m(m-1)}{2}}\prod_{l=0}^{m-n-1}\prod_{j=1}^{n}\sin\alpha(l+2j)h\cdot\prod_{j=1}^{m-n}\sin^{m-n-j}(\alpha jh)\cdot\prod_{j=1}^{n}\sin^{n-j}(2\alpha jh). \tag{27}$$

(b) $m \geqslant 2, n = 0$

$$\delta_m(x,0) = (-4)^{\frac{m(m-1)}{2}}\prod_{j=1}^{m-1}\sin^{m-j}(\alpha jh)\prod_{k=-m+1}^{m-1}\sin^{[\frac{m+1-|k|}{2}]}\alpha(x+kh). \tag{28}$$

(c) $m \geqslant 2, n = m$

$$\delta_m(x,m,\alpha) = \delta_m(x,0,2\alpha). \tag{29}$$

Proof. We present here only a hint of the proof. The complete proof can be found in a forthcoming paper [14]. The idea of the proof is to first establish a recursion relation of the form

$$\delta_{m-k}(x,n) = 2^{2(m-k-1)}(-1)^{m-k-1}\prod_{j=1}^{m-k}\sin(\alpha(x-(m-k+1-2j)h))\cdot\prod_{j=1}^{m-k-n+1}\sin(\alpha jh)$$
$$\cdot\prod_{j=1}^{n}\sin(\alpha(m-k-n-1+2j)h)\cdot\delta_{m-k-1}(x,n),$$

for $0 \leqslant k \leqslant m-n-1$.

[10] In fact, Darboux established a similar property for a more special case of the Sturm–Liouville equation for the case $m = 1$. Its extensions to the case of linear and nonlinear PDEs of any order and their difference and non-Abelian versions were proposed by Matveev in 1979 [17–19].

[11] Here, we use the standard notation $[x]$ for the integer part of x and $|x|$ for its absolute value.

A repeated application of this recursion relation leads to the formula

$$\delta_m(x,n) = \prod_{k=n+1}^{m} 2^{2(k-1)}(-1)^{k-1}\prod_{j=1}^{k}\sin\alpha(x-(k+1-2j)h)\prod_{j=1}^{n}\sin\alpha(k-n-1+2j)h$$
$$\cdot\prod_{k=n+2}^{m}\prod_{j=1}^{k-n-1}\sin(\alpha jh)\cdot\delta_n(x,n). \qquad (30)$$

But it is obvious that $\delta_n(x,n,\alpha) = \delta_n(x,0,2\alpha)$. So the computation of $\delta_m(x,n)$ is reduced to calculate $\delta_n(x,0)$, which is trivial to do, using formula (15) already mentioned above.

Therefore, combining (30) and (15), we get (27) which completes the proof. □

3.3. Applications to the DDPT-I model

The integrability of DDPT-I was proved 'à la Darboux' in [22, 23] exploring the factorized representation (26) of the general solution and the following statement.

Proposition 3.1. *The function*

$$F_1(x,m,n) = \prod_{k=0}^{n} c(x-kh)\prod_{j=0}^{m} s(x-jh) \qquad (31)$$

satisfies the DDPT-I equation with $\lambda = 2\cos\alpha h(n+m+2)$.

Here, we present the following new result.

Theorem 3.3. *The DPT potentials* $v_{mn}(x)$ *defined in (18) can be represented as a ratio of four Casorati determinants:*

$$v_{mn}(x) = \frac{\delta_m(x-h,n)\delta_m(x+2h,n)}{\delta_m(x+h,n)\delta_m(x,n)}. \qquad (32)$$

The general solution of the DDPT-I equation is given by the formula

$$\psi(x,\lambda) = \frac{\delta_{m+1}[f,f_1,\ldots,f_m](x,n)}{\delta_m[f_1,\ldots,f_m](x+h,n)}, \qquad (33)$$

where f *satisfies* $f(x+h)+f(x-h) = \lambda f$ *and* $f_j(x)$ *are defined in (4).*

Hint of the proof. Using formula (27), we obtain that

$$\frac{\delta_{m+1}(x,n+1)}{\delta_m(x+h,n)} = k_{mn}\cdot F_1(x,m,n),$$
$$k_{mn} = 2^{2m+n+1}(-1)^m\prod_{l=0}^{m-n-1}\sin(\alpha h(l+2n+2))\prod_{j=1}^{n}\sin(2\alpha jh), \qquad (34)$$

where k_{mn} does not depend on x. This proves formula (32)[12].

Now, (33) becomes obvious from (21) and (24). □

Thus, we got the new proof of the integrability of the DDPT-I model.

Indeed, the representation (32) for the DDPT-I potential has the same advantage as the Wronskian representation of the DPT potentials: it allows us to automatically obtain a solution to the Cauchy problem for the difference KdV hierarchy with DDPT-I potentials taken as initial data. Here we skip a further discussion of this point with respect to volume limitations. The details can be found in our recent article [14], where it is also briefly explained how one can obtain the main results of the first section by just taking (after an appropriate rescaling) the limit $h \to 0$ in the formulae of this section.

[12] This also proves proposition 2.1 (31) in a way which is shorter and easier with respect to [22, 23].

4. Casorati determinant representation of the DDPT-II model

We consider in this section another kind of difference deformation of DPT potentials connected with a difference equation:

$$w(x)f(x+h)+f(x-h)+b(x)f(x)=\lambda f(x). \tag{35}$$

Let $w = w_{mn}$ and $b = b_{mn}$ be the potentials given by

$$w_{mn}(x)=\frac{c(x-nh/2)c(x-(n-1)h/2)c(x+(n+1)h/2)c(x+(n+2)h/2)}{c(x)c(x+h)(c(x+h/2))^2}\cdot\frac{s(x-mh/2)s(x-(m-1)h/2)s(x+(m+1)h/2)s(x+(m+2)h/2)}{s(x)s(x+h)(s(x+h/2))^2} \tag{36}$$

and

$$b_{mn}(x)=-\frac{2s(mh/2)s((m+1)h/2)c(nh/2)c((n+1)h/2)}{c(x-h/2)c(x+h/2)}-\frac{2c(mh/2)c((m+1)h/2)s(nh/2)s(n+1)h/2)}{s(x-h/2)s(x+h/2)}. \tag{37}$$

We call (35) with $w(x) = w_{mn}(x)$ and $b(x) = b_{mn}(x)$ the DDPT-II equation (difference Darboux–Pöschl–Teller-II equation).

In (36) and (37), m, n are non-negative integers, α is an arbitrary scaling parameter, $c(x) := \cos(\alpha x)$ and $s(x) := \sin(\alpha x)$.

Integrability of the DDPT-II equation by means of the elementary functions was first established in [13] in a way similar to [22, 23].

Passing to the limit $h \to 0$, we restore the differential DPT equation (see [13] for details).

Similar to the case of the DPT potentials and the DDPT-I model, the DDPT-II equation is invariant under the action of each of the maps $m \to -m-1$ and $n \to -n-1$.

4.1. Modified difference Darboux transformation and the DDPT-II equation

Here again, we use some special case of the results proved in [18, 19].

The difference Darboux transformation $\psi_1(x)$ of an arbitrary solution $f(x,\lambda)$ of (35), generated by the fixed solution $f_1(x)$ of the same equation (35) with $\lambda = \lambda_1$, is defined by the formula

$$\psi_1(x)=\frac{\begin{vmatrix} f(x-h/2) & f_1(x-h/2)\\ f(x+h/2) & f_1(x+h/2)\end{vmatrix}}{f_1(x)},$$
$$\psi_1(x)=f(x-h/2)-\sigma_1(x)f(x+h/2)$$
$$\sigma_1(x)=\frac{f_1(x-h/2)}{f_1(x+h/2)}. \tag{38}$$

Theorem 4.1. *$\psi_1(x)$ represents a general solution to the following equation:*

$$w_1(x)\psi_1(x+h)+\psi_1(x-h)+b_1(x)\psi_1(x)=\lambda\psi_1(x), \tag{39}$$

$$w_1(x)=\frac{f_1(x+3h/2)f_1(x-h/2)}{(f_1(x+h/2))^2}v(x+h/2), \tag{40}$$

$$b_1(x)=b(x-h/2)+\frac{f_1(x-3h/2)}{f_1(x-h/2)}-\frac{f_1(x-h/2)}{f_1(x+h/2)}. \tag{41}$$

Proof. See for instance [18, 19].

The Darboux transform maps (35) into an equation of the same form, with the same value of spectral parameter λ, but with a new potential constructed in terms of the initial potential $v(x)$ and a fixed solution $f_1(x)$ of (35)[13]. □

4.2. Multiple difference Darboux transform 2

We use the following notations:

$$d_m(x) := d_m[f_1, f_2, \ldots, f_m](x)$$
$$= \begin{vmatrix} f_1(x-(m-1)h/2) & f_2(x-(m-1)h/2) & \ldots & f_m(x-(m-1)h/2) \\ f_1(x-(m-3)h/2) & f_2(x-(m-3)h/2) & \ldots & f_m(x-(m-3)h/2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x+(m-1)h/2) & f_2(x+(m-1)h/2) & \ldots & f_m(x+(m-1)h/2) \end{vmatrix}, \quad (42)$$

$$c_m(x) := c_m[f_1, \ldots, f_m](x) :=$$
$$:= \begin{vmatrix} f_1(x-(m-1)h/2) & f_2(x-(m-1)h/2) & \ldots & f_m(x-(m-1)h/2) \\ f_1(x-(m-5)h/2) & f_2(x-(m-5)h/2) & \ldots & f_m(x-(m-5)h/2) \\ f_1(x-(m-7)h/2) & f_2(x-(m-7)h/2) & \ldots & f_m(x-(m-7)h/2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x+(m+1)h/2) & f_2(x+(m+1)h/2) & \ldots & f_n(x+(m+1)h/2) \end{vmatrix}. \quad (43)$$

The last determinant has an 'almost Casoratian' structure, i.e. a strictly Casoratian structure for the last $m-1$ rows, and a double shift of the arguments of f_j between the first row and the second row.

Let f_j, $j = 1, \ldots, m$, be the solutions of the difference equations:

$$w(x) f_j(x+h) + f_j(x-h) + b(x) f_j(x) = \lambda_j f(x). \quad (44)$$

Theorem 4.2. *The function* $\psi_m(f)(x)$*,*

$$\psi_m(f)(x) := \frac{d_{m+1}[f, f_1, \ldots, f_m](x)}{d_m[f_1, \ldots, f_m](x+h/2)}, \quad (45)$$

gives a general solution to the following equation:

$$w_m(x)\psi_m(f)(x+h) + \psi_n(f)(x-h) + b_m(x)\psi_m(f)(x) = \lambda\psi_n(f)(x), \quad (46)$$

for

$$w_m(x) := \frac{d_m(x-h/2) d_m(x+3h/2)}{d_m(x+h/2)^2} w(x+mh/2), \qquad m \geqslant 1, \quad (47)$$

and

$$b_m(x) := b(x-mh/2) + \frac{c_m(x-3h/2)}{d_m(x-h/2)} - \frac{c_m(x-h/2)}{d_m(x+h/2)}, \qquad m \geqslant 2, \quad (48)$$

Formula (47) is well known (see [18, 19]). Formula (48) can be easily proved in two steps: first, by establishing the functional difference analogue of the second line in (5.1.26) in [18]

[13] In contrast to (35), equation (17) is not covariant with respect to the action of the Darboux transformation described in this section, since it does not preserve the coefficient $b(x)$.

and, second, by eliminating the t-derivative with the help of an auxiliary evolution equation for the Toda lattice. Setting in the obtained formula $t = 0$, we can easily prove (48). Here we skip the related details for brevity although every reader familiar with the Darboux transformation method can easily reconstruct the details which will be given in our forthcoming article [15].

Similar to the DDPT-I equation function $\psi_n(x)$ can also be represented in the following factorized form:

$$\psi_m(x) = (U^{-1} - \sigma_n(x)U)\ldots(U^{-1} - \sigma_1(x)U)f(x,\lambda),$$
$$U^{\pm k}f(x) := f(x \pm kh/2), \quad \sigma_j(x) := \frac{\Psi_j(x - h/2)}{\Psi_j(x + h/2)},$$
$$\Psi_0(x) := f(x), \Psi_j(x) := \psi_{j-1}(x)|_{f(x)=f_j(x)}, j = 1,\ldots,m.$$

4.3. *Explicit formulae for some special determinants*

Here we show that taking $w(x) = 1$, $b(x) = 0$ and defining the functions $f_1,\ldots,f_n$ as in (4), we get from formulae (47), (48), (41) the same potentials w_{mn} and b_{mn} as defined by (36) and (37) respectively.

Let n and d be some non-negative integers, $m = n + d$, h, α be some real parameter. The main technical tools we use are explicit formulae for the determinants $D_{mn}(x)$ and $C_{mn}(x)$ defined as follows.

We denote the determinant with D_{mn}:

$$\begin{aligned} D_{mn} &= D_{mn}(x,h) = D_{mn}[f_1,\ldots,f_m](x) \\ &:= \det(f_j((x - (m+1-2i)h/2)))_{i,j=1,\ldots,m}). \end{aligned} \tag{49}$$

It is clear that

$$D_{mn}(x,h) = \delta_{mn}(x,h/2), \tag{50}$$

where δ_{mn} is defined by (1).

$C_{mn}(x,h) = c_m(f_1,\ldots,f_m)$, where $f_j(x)$ are defined in (4). The determinant $C_{mn}(x,h)$ can also be explicitly computed.

Theorem 4.3.

(i) $m \geqslant 2$, $1 \leqslant n \leqslant m-1$

$$\begin{aligned} C_{mn} = a_{mn} \cdot &\prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]}\alpha(x+(k+2)h/2) \prod_{k=-n+1}^{n-1} \cos^{[\frac{n+1-|k|}{2}]}\alpha\alpha(x+(k+2)h/2) \\ &\cdot \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\sin\alpha(x+h)\cos\alpha(x+h)} \\ &\cdot (\sin(\alpha(m+n+1)h/2)\cos\alpha(x-(n-2)h/2) \\ &+ \sin(\alpha(m-n-1)h/2)\cos\alpha(x+(n+2)h/2)), \\ a_{mn} = 2^{\frac{n(n+1)}{2}}(-4)^{\frac{m(m-1)}{2}} &\prod_{l=0}^{m-n-1}\prod_{j=1}^{n}\sin\alpha(l+2j)h/2 \\ &\cdot \prod_{j=1}^{m-n}\sin^{m-n-j}(\alpha jh/2) \cdot \prod_{j=1}^{n}\sin^{n-j}(\alpha jh). \end{aligned} \tag{51}$$

(ii) $m \geqslant 2, n = 0$

$$C_{m0} = (-1)^{\frac{m(m-1)}{2}} (2)^{\frac{m(m-3)}{2}+1} \prod_{j=1}^{m-1} \sin^{m-j}(\alpha j h/2) \prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]} \alpha(2x+(k+2)h) \cdot \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\sin\alpha(x+h)} \cdot (\sin(\alpha m h/2)\cos(\alpha h/2)). \qquad (52)$$

(iii) $m \geqslant 2, n = m$

$$C_{mm} = (-1)^{\frac{m(m-1)}{2}} (2)^{m(m-1)} \prod_{j=1}^{m-1} \sin^{m-j}(\alpha j h) \cdot \prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]} \alpha(2x+(k+2)h) \cdot \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\sin\alpha(x+h)\cos\alpha(x+h)} \cdot (\sin(\alpha(2m+1)h/2)\cos(\alpha(x-(m-2)h/2) - \sin(\alpha h/2)\cos\alpha(x+(m+2)h/2)). \qquad (53)$$

Proof. It is enough to check the following relation:

$$C_{mn}(x) = \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\cos\alpha(x+h)\sin\alpha(x+h)} \cdot (\sin(\alpha(m+n+1)h/2)\cos\alpha(x-(n-2)h/2) + \sin(\alpha(m-n-1)h/2)\cos\alpha(x+(n+2)h/2))D_{mn}(x+h). \qquad (54)$$

The latter formula can be proved by using the recursion relation

$$C_{m+1n+1}(x) = \frac{D_{m+1n+1}(x)}{D_{mn}(x+h/2)} D_{mn}(x+3h/2) + \frac{C_{mn}(x-h/2)}{D_{m,n}(x+h/2)} D_{m+1n+1}(x+h). \qquad (55)$$

We postpone further details for a more detailed publication. □

4.4. Getting the DDPT-II model from multiple Darboux dressing formulae

Integrability of the DDPT-II model was first proved in [13] using an approach similar to [22, 23]. Below, we present a different and shorter proof.

Theorem 4.4. *The DPT potentials $w_{mn}(x)$ and $b_{mn}(x)$ defined by (36) and (37) can be expressed in terms of Casorati determinants in the following way:*

$$w_{mn} = \frac{D_{mn}(x-h/2)D_{mn}(x+3h/2)}{(D_{mn}(x+h/2))^2}, \qquad (56)$$

$$b_{mn}(x) = \frac{C_{mn}(x-3h/2)}{D_{mn}(x-h/2)} - \frac{C_{mn}(x-h/2)}{D_{mn}(x+h/2)}. \qquad (57)$$

The general solution of the DDPT-II equation can be written as

$$\phi(x,\lambda) = \frac{D_{m+1,n+1}[f, f_1, \ldots, f_m](x)}{D_{mn}[f_1, \ldots, f_m](x+h/2)}. \qquad (58)$$

Hint of the proof. It is sufficient to replace D_{mn} and C_{mn} by their expressions (49) and (51) respectively.

Using the explicit formula of D_{mn}, it is easy to see that[14]

$$\frac{D_{m+1,n+1}(x)}{D_{mn}(x+h/2)} = p_{mn} \cdot \prod_{k=0}^{m} \sin(x - kh/2) \prod_{k=0}^{n} \cos(x - kh/2) \equiv p_{mn} \cdot \Phi_1(x, m, n),$$

$$p_{mn} := 2^{2m+n+1}(-1)^m \prod_{l=1}^{m-n} \sin(\alpha(m+n+2-l)h/2)) \cdot \prod_{j=1}^{n} \sin(\alpha jh).$$

The above formula shows that (56) really holds.

Using an explicit formula for D_{mn} and (51), we can also easily check that (57) holds. □

4.5. *DDPT-II equation and Askey–Wilson functions*

The Askey–Wilson spectral problem reads as

$$A(z)A(q^{-1}z^{-1})\psi(qz) - (A(z) + A(z^{-1}) - \tilde{a} - \tilde{a}^{-1})\psi(z) + \psi(q^{-1}z) = (\lambda + \lambda^{-1})\psi(z),$$
$$A(z) := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{\tilde{a}(1-z^2)(1-qz^2)}, \qquad \tilde{a} := \sqrt{q^{-1}abcd}. \tag{59}$$

Taking

$$a = e^{(n+1)2ih}, \qquad b = -e^{(m+1)2ih}, \qquad c = e^{(n+2)2ih},$$
$$d = -e^{(m+2)2ih}, \qquad q = e^{4ih}, \qquad z = e^{2ix}, \tag{60}$$

it is easy to check that in this case the Askey–Wilson equation (59) becomes the DDPT-II equation. It is well known [16] that the general solution to the Askey–Wilson equation can be written in terms of ${}_4\Phi_3$ q-hypergeometric functions or by means of very well-poised ${}_8\phi_7$ basic hypergeometric series. In the particular case of the DDPT-II equation, the general solution is expressed by means of the Casorati determinants composed from the elementary functions. Therefore, our results lead to the following remarkable fact.

Let a, b, c, d be chosen as in (60). Then the related Askey–Wilson functions become the elementary functions which are described by means of Casorati determinants using theorem 4.4.

We obtain a similar result for the hyperbolic version of the DDPT equation writing e *instead of* e^{i} *in (60). Of course, this statement merits further precision in the spirit of works [8–10]. We will describe the related explicit evaluation formulae for the q-hypergeometric functions in a more detailed publication.*

5. Concluding remarks

- Theorem 2.2 proves that the DPT trigonometric potentials produce a subfamily of two-dimensional Huygens potentials $V(r, \varphi)$ via a formula

$$V(r,\varphi) = \frac{1}{r^2}\left[\frac{m(m+1)}{\sin^2\varphi} + \frac{n(n+1)}{\cos^2\varphi}\right], \tag{61}$$

[14] It was shown in [13] in a different and longer way that $\Phi(x, m, n)$ solves the DDPT-II equation with $\lambda = 2\cos(\alpha h(m+n+2))$. Here, we get the same result as a trivial consequence of the generic Casorati determinant formalism described above.

since the generic 2D Huygens potentials in polar coordinates are described by the formula[15]

$$V(r,\varphi) = -\frac{2}{r^2}(\partial_\varphi)^2 \log W(\chi_1,\ldots,\chi_m),$$
$$\chi_j = \sin(k_j\varphi + \delta_j), \qquad j = 1,\ldots,m; \quad k_j \in \mathbb{Z}, \quad \delta_j \in \mathbb{R}.$$

Therefore, taking $k_j = a_j$, where a_j are the same as in theorem 2.2 and, taking $\delta_j = 0$, we get the aforementioned link between the DPT potentials and Huygens potentials. Moreover, the related remarkable formula (5) for the Hadamard coefficients $U_\nu(x,\xi)$ of [1] can be essentially simplified for the special Huygens potentials of form (61) using theorem 2.2.

- We have also shown that scepticism expressed in [28, p 371], concerning the use of generic Casorati determinant formulae for studying discrete DPT potentials, was not justified since the case considered in [28] corresponds to a special reduction of the DDPT-I or DDPT-II equation, namely to the case $n = 0$, and $x \in \mathbb{Z}, h = 1$: we can easily recover all the results obtained in [28] using the results of this work.
- The hyperbolic case of DDPT-I and DDPT-II models is quite similar: we have to replace α by $i\alpha$. Of course, the related spectral properties are different but the algebraic structure of the main formulae remains the same. We postpone the detailed discussion of the hyperbolic DPPT-I and DDPT-II equations in the spirit of this work for more detailed publication. The related bound-state eigenfunctions were described in [13, 22, 23].
- It is interesting to mention that the same expression as determinant (15) (up to a different constant normalization factor) recently appeared in [30, theorem A, theorem 5.1] as a density of probability measure describing the asymptotic distribution of the Frobenius roots on the m-dimensional Abelian varieties over finite fields F_α when $\alpha \to \infty$.
- Hirota or Sato-like formulae for the solutions of some of the equations considered (also providing solutions by means of the elementary functions) were obtained via the IST approach by Kirillov and van Diejen, first, for the DPT equation [8, 9] and, next, for the DDPT-I equation with $n = 0$ [10] where solutions in terms of q-hypergeometric functions also have been constructed. We claim that it is possible to obtain these formulae via taking an appropriate limit of some finite-gap Baker–Akhiezer functions. We postpone the detailed explanation for future publications.
- The content of section 2 is obviously connected with the usual (trigonometric) two-particle quantum BC_1 Calogero–Moser system. The results of sections 3 and 4 are obviously relevant to the trigonometric case of some two-particle quantum difference Calogero–Moser systems, although we avoided reducing our L-operators to CM Hermitian Hamiltonians by well-known similarity transformations. See [11] for the related definitions and detailed comments.
- After submission of this article, we were informed by Professor Sasaki, on 16 February 2009, about his work with S Odake which has some intersection points with our work. Somehow our results are essentially different from [24].

Note added in proof. As we observed at the last moment, this connection was first announced in the work [2] (p 116, remark II) without proof, which was replaced by the phrase 'it is easy to see that'. The proof which we found is somehow much more difficult with respect to a proof of generic Crum dressing formula.

[15] In [1–3] the formula below was written in a slightly different but strictly equivalent form, using the Wronskian of $m + 1$ cosine functions.

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